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2002 J. Phys. A: Math. Gen. 35 993

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INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 993–999 PII: S0305-4470(02)25887-4

Analytical solution of the dynamical spherical MIT bag

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Received 18 June 2001, in final form 8 November 2001
Published 18 January 2002
Online at stacks.iop.org/JPhysA/35/993

Abstract

We prove that when the bag surface is allowed to move radially, the equations of motion derived from the MIT bag Lagrangian with massless quarks and a spherical boundary admit only one solution, which corresponds to a bag expanding at the speed of light. This result implies that some new physics ingredients, such as coupling to meson fields, are needed to make the dynamical bag a consistent model of hadrons.

PACS number: 12.39.Ba

The MIT bag model, in which hadrons are modelled by the states formed with free quarks confined inside an impenetrable bag, has been rather successful in reproducing static ground state properties of hadrons [1]. Because of its simplicity, especially its spherically symmetric version, the model has been used extensively in the discussion of various phenomena ranging from strange stars [2] to ultra-relativistic heavy-ion collisions [3], even though these often involve situations of high density/temperature, where the applicability of the model is doubtful. Many attempts have also been made to augment the basic MIT bag model with new ingredients, such as the partial restoration of chiral symmetry via meson coupling [4] and inclusion of perturbative gluon exchanges among quarks [5]. However, in almost all discussion, the hadron bag is treated as a static boundary between the perturbative and nonperturbative vacua, and excitations of the hadron are associated with the quark degree of freedom. The few notable exceptions [6–13], which allowed for the possibility of a dynamical bag boundary, focused mainly on reproducing the correct phenomenological parity order of the low-lying states of the nucleon, but several approximations and modifications to the theory had to be employed. Rebbi's idea [6] was to perform an approximate quantization of the small oscillations about the lowest-energy, spherically symmetric solution of the MIT bag model. In order to implement it he used the Dirac method for the quantization of systems with constraints. Hasenfratz and Kuti [7] bypassed the difficulties of a constrained system by adding a surface tension to the bag and hence providing a kinetic term for the bag's surface. They then quantized the bag's motion using the adiabatic approximation. In the same spirit the authors of [8–10]

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introduced an effective surface tension and applied it to a model which includes a quark–pion interaction term at the boundary. Fiebig [11] instead used a quite different approach; he obtained approximate classical solutions for the bag's radius and the conjugate momentum and quantized the system with the Bohr–Sommerfeld quantization prescription. Guichon [12] and Zhang [13] both derived approximate classical solutions for the fields in a non-static cavity, using the small oscillation and adiabatic approximations respectively. They then quantized the solutions.

Our approach is complementary to those introduced above. All these works showed unequivocally that taking into account the surface motion can resolve the problem that in the static model, the P-state energies are too low and the first excited S states are too high. This appears to be a general characteristic of bag models, regardless of the specific features of each model. It is necessary at this point to study the exact dynamical solutions of a particular model. In particular the question of whether a dynamical bag model can consistently describe hadrons was not addressed. This is particularly evident from the fact that all the previous works on the non-static MIT bag bypassed the problem of exactly satisfying the linear boundary condition. The motivation of the present work is to address these issues. In this paper we discuss, analytically and without approximations, the consequences of allowing the bag boundary to move.

We consider the basic MIT bag model with free fermions inside a spherically symmetric but non-static bag. We shall show that the equations of motion require that the fermion field $\psi(t, r)$ vanishes at the bag boundary $r = R(t)$. Hence our problem reduces to a quite general one: that of the Dirac equation in a spherical dynamical cavity with $\psi(t, R(t)) = 0$. Furthermore, we shall show that for massless fermions the *only* solution is that of a bag expanding with the speed of light. From this unexpected result, which evidently has no phenomenological application, we infer that some new physics ingredients, such as an interaction term with other fields, have to be introduced to make the more general dynamical bag model consistent and physical [14]. In the case of massive quarks we could not find a solution and we conjecture that in fact it does not exist.

The MIT bag Lagrangian density [1] is written as

$$\mathcal{L} = \left[\frac{\mathrm{i}}{2}\left(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\gamma^{\mu}\psi\right) - B\right]\theta_{v}(x) - \frac{1}{2}\bar{\psi}\psi\Delta_{\mathrm{s}} \tag{1}$$

where $\theta_v(x)$ is unity inside the bag and zero outside and

$$\frac{\partial\theta_v}{\partial x^{\mu}} = n_{\mu}\Delta_{\mathrm{s}} \tag{2}$$

Δ_{s} being the surface delta-function and n_μ the normal vector to the bag. From the Euler–Lagrange equation of motion

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\bar{\psi})} = 0 \tag{3}$$

we obtain

$$\mathrm{i}\gamma^{\mu}\partial_{\mu}\psi = 0 \qquad \text{inside the bag} \tag{4}$$

$$\mathrm{i}\gamma^{\mu}n_{\mu}\psi = \psi \qquad \text{on the surface.} \tag{5}$$

This last equation may be considered as the boundary condition for equation (4). Energy–momentum conservation implies a further constraint at the boundary [1, 15]:

$$Bn^{\nu} = \tfrac{1}{2}[\partial^{\nu}(\bar{\psi}\psi)]_{r=R}. \tag{6}$$

We now look for a spherically symmetric solution of the above equations. In this case we have $\theta_v = \theta(R-r)$, $\Delta_s = \delta(R-r)$ and

$$n_\mu = (\dot{R}, -\hat{r}). \tag{7}$$

We first find an explicit expression for the boundary condition. Equation (5) becomes

$$\mathrm{i}\dot{R}\gamma^0\psi - \mathrm{i}\vec{\gamma}\cdot\hat{r}\psi = \psi \tag{8}$$

with

$$\vec{\gamma}\cdot\hat{r} = \begin{pmatrix} 0 & \vec{\sigma}\cdot\hat{r} \\ -\vec{\sigma}\cdot\hat{r} & 0 \end{pmatrix}. \tag{9}$$

We can write the spinor ψ as [15, 16]

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} g(r,t)\mathcal{Y}_{jl}^{j_3} \\ \mathrm{i}f(r,t)\mathcal{Y}_{jl'}^{j_3} \end{pmatrix} \tag{10}$$

where $\mathcal{Y}_{jl}^{j_3}$ contains the spin and angular parts of the wavefunction. Observing that $(\vec{\sigma}\cdot\hat{r})\mathcal{Y}_{jl}^{j_3} = -\mathcal{Y}_{jl'}^{j_3}$ [15], we can write equation (8) as

$$\begin{aligned} &\mathrm{i}\dot{R}g(t,R) - f(t,R) = g(t,R) \\ &\dot{R}f(t,R) - \mathrm{i}g(t,R) = \mathrm{i}f(t,R). \end{aligned} \tag{11}$$

If $\dot{R} = 0$ we have the familiar boundary condition for the static MIT bag, i.e. $g(t,R) = -f(t,R)$, which also corresponds to that in the Bogolioubov model [16], and the analytical solutions are well known. However it is easy to verify that, if $\dot{R} \neq 0$, then equations (11) can be satisfied only if

$$g(t,R) = f(t,R) = 0. \tag{12}$$

Notice from equation (6) that this implies $B = 0$, and in this way energy–momentum is conserved regardless of the motion of the bag. This also means that equation (6) does not provide information about $\dot{R}$.

Performing the change of variable $y = rR_0/R(t)$ one could recast the problem into a static boundary one. In this framework the motion of the bag's surface is treated as a time-dependent perturbation to the static Hamiltonian and one looks for the solutions by means of a time-dependent expansion in terms of the static cavity eigenfunctions, $\exp\{-\mathrm{i}E_n t\}\psi_n(y)$ [18]. However in our case this approach cannot provide us the solution. In fact, writing the wavefunction ψ_f for the fixed boundary problem as

$$\psi_f = \sum_{n=0}^{\infty} c_n(t)\mathrm{e}^{-\mathrm{i}E_n t}\psi_n(y) \tag{13}$$

we can in general work out $c_n(t)$, for example by perturbation theory, but it is well known that the static eigenfunctions, $\exp\{-\mathrm{i}E_n t\}\psi_n(y)$, of the Dirac field inside a spherical cavity are non-zero at the boundary [15], and we have no way to impose the boundary condition equation (12) on expression (13). In other words, although at the initial time we can always choose a suitable combination of $\psi_n(y)$ which is zero at $y = R_0$, at subsequent times $\psi_f(t, R_0)$, as expressed in equation (13), will in general be different from zero. We hence need to proceed in a different way.

Substituting expression (10) for ψ in equation (4), with $l = 0$ and $l' = 1$ in order to have spherical symmetry, we obtain two coupled equations:

$$-\mathrm{i}\frac{\partial g}{\partial t} = \frac{\partial f}{\partial r} + \frac{2}{r}f \tag{14}$$

$$\mathrm{i}\frac{\partial f}{\partial t} = \frac{\partial g}{\partial r}. \tag{15}$$

Integrating equation (15) with respect to time and substituting the expression for $f(t, r)$ in equation (14) we have

$$f = A(r) - \mathrm{i} \int \mathrm{d}t \, \frac{\partial g}{\partial r} \tag{16}$$

$$-\mathrm{i}\frac{\partial g}{\partial t} = -\mathrm{i} \int \mathrm{d}t \left[\frac{\partial^2 g}{\partial r^2} + \frac{2}{r}\frac{\partial g}{\partial r} \right] + \frac{\mathrm{d}A\,(r)}{\mathrm{d}r} + \frac{2}{r}A(r) \tag{17}$$

where $A(r)$ is a time-independent function to be determined. Equation (17) is the spherical wave equation in integro-differential form, whose general solution can be written in the form

$$g(t, r) = \frac{1}{r}\,[G(t-r) - G(t+r)] + Ct + D \tag{18}$$

with C and D two constants. In principle, a term of the form α/r is allowed in equation (18). However, it makes the wavefunction not normalizable and unphysical. We have also checked that this term does not affect at all our proof. We therefore set $\alpha = 0$.

Inserting the above expression in equation (17) yields an equation for $A(r)$:

$$\frac{\mathrm{d}A}{\mathrm{d}r} = -\frac{2}{r}A - \mathrm{i}C \tag{19}$$

whose solution is

$$A(r) = -\frac{\mathrm{i}}{3}Cr. \tag{20}$$

Hence we can write an explicit expression for f as

$$f(t, r) = \frac{\mathrm{i}}{r}\left\{ G(t-r) + G(t+r) + \frac{1}{r}\,[Q(t-r) - Q(t+r)] \right\} - \mathrm{i}\frac{C}{3}r \tag{21}$$

with $Q'(z) = G(z)$. Now using the boundary conditions equation (12) we can derive the relations among $G(t-R)$, $G(t+R)$, $R(t)$ and $\dot{R}(t)$. Equating $g(t, R)$ and $f(t, R)$ to zero we have

$$g(t, R) = \frac{1}{R}\,[G(t-R) - G(t+R)] + Ct + D = 0 \tag{22}$$

$$f(t, R) = \frac{\mathrm{i}}{R}\left\{ G(t-R) + G(t+R) + \frac{1}{R}[Q(t-R) - Q(t+R)] \right\} - \mathrm{i}\frac{C}{3}R = 0. \tag{23}$$

Taking the time derivative of both equations we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}g(t, R) = {} & \frac{1}{R}\{(1-\dot{R})G'(t-R) - (1+\dot{R})G'(t+R)\} \\ & - \frac{\dot{R}}{R^2}\,[G(t-R) - G(t+R)] + C = 0 \end{aligned} \tag{24}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}f(t, R) = {} & -\frac{\dot{R}}{R}f(t, R) - \frac{\mathrm{i}}{R}\left\{ \frac{2}{3}CR\dot{R} - (1-\dot{R})G'(t-R) - (1+\dot{R})G'(t+R) \right. \\ & + \frac{\dot{R}}{R^2}\,[Q(t-R) - Q(t+R)] \\ & \left. + \frac{1}{R}\left[(\dot{R}-1)G(t-R) + (\dot{R}+1)G(t+R) \right] \right\} = 0. \end{aligned} \tag{25}$$

From the last two equations, and by means of equations (22) and (23), we can find

$$(1-\dot{R})G'(t-R) - (1+\dot{R})G'(t+R) + \dot{R}(Ct + D) + CR = 0 \tag{26}$$

and

$$-(1-\dot{R})G'(t-R)-(1+\dot{R})G'(t+R)+Ct+D+CR\dot{R}=0. \qquad (27)$$

These two equations contain all the information we need to solve the problem, i.e. to find $\dot{R}(t)$ for any $t > t_0$ and $G(z)$ for any $z > t_0 + R(t_0)$. In fact, using the same argument as in [17], as long as $|\dot{R}| \leqslant 1$, at each time $t \geqslant t_0$ $G'(t-R)$ is known and we have two equations for the two unknowns $G(t+R)$ and $\dot{R}(t)$. By summing and subtracting equations (26) and (27) we can decouple $\dot{R}$ and $G'(t+R)$ as follows:

$$(1+\dot{R})[-2G'(t+R)+(Ct+D+CR)]=0 \qquad (28)$$

$$(1-\dot{R})[2G'(t-R)-(Ct+D-CR)]=0. \qquad (29)$$

It is finally evident that equation (29) implies

$$\dot{R}(t)=1 \qquad (30)$$

and from equation (28) we have

$$G'(t+R(t))=\tfrac{1}{2}\left[Ct+D+CR(t)\right]. \qquad (31)$$

Notice that $\dot{R}=-1$ is not allowed because equation (29) would not be satisfied. Since expressions (18) and (21) represent the general solution of the problem, the above result excludes the possibility of any other solution. It is also important to note that, with $\dot{R}=1$, $t-R(t)=t_0-R(t_0)$.

Now, knowing that $R(t)=R_0+t-t_0$ and defining $z \equiv t+R(t)$, we have

$$G'(z)=\tfrac{1}{2}(Cz+D) \qquad z \geqslant t_0+R_0. \qquad (32)$$

At this point, in order to have a clear understanding of the solution, we set $C=D=0$. In fact, the two constants, with the boundary conditions given by equations (22) and (23), are physically irrelevant. We obtain

$$G(z)=G(z_0)=G(t_0-R_0) \qquad z \geqslant t_0+R_0=z_0 \qquad (33)$$

$$Q(z)=G(z_0)(z-z_0)+Q(z_0) \qquad z \geqslant t_0+R_0=z_0. \qquad (34)$$

Given $\psi(t_0,r)$, which for example is different from zero at some r, then $\psi(t,r)$ will go to zero as $t \to t_0+R_0+r$, and so we shall have

$$\psi(t,r)=0 \qquad r \leqslant t-t_0-R_0 \qquad r \geqslant t-t_0+R_0. \qquad (35)$$

Therefore, after a time $t=R_0+t_0$, the solution represents an expanding spherical shell with internal radius $R_{\text{in}}=t-t_0-R_0$ and external radius $R_{\text{out}}=t-t_0+R_0$. The solution for $t<t_0$ can be found analogously, evolving backward in time. In this case equations (28) and (29) imply $\dot{R}=-1$ and $G'(t-R)$ would be the unknown. Obviously the solution becomes singular as $R(t)=0$.

This unexpected solution of the massless Dirac equation in a spherical dynamical cavity is evidently due to the boundary conditions equation (12). However, as far as one considers the Dirac field completely confined in a cavity, any other boundary condition would violate unitarity, as can be checked easily by taking the time derivative of the norm of the field.

From a physical point of view the problem lies in the fact that the MIT bag model sets the field to zero outside the bag already in the Lagrangian. In the Bogolioubov model with a finite square-well potential, there is a non-zero field also outside the well. If the wall moves inward, the field gains enough energy so that parts of it can go out of the well. The higher the potential the more energy is transferred to the field during a compression, mainly because the Dirac field at the wall does not approach zero as the potential goes to infinity.

The MIT bag model overcomes this problem by means of the boundary condition (equation (12)), which sets the field to zero at the moving boundary. The drawback, as we have seen, is the absence of a baglike solution for a dynamical boundary. In relation to our results it is particularly interesting to note that Fiebig's calculations [11] imply that the MIT bag model does not allow moving boundary solutions if the initial state is the lowest static solution. Not using exactly the linear boundary condition equation (8), Fiebig did obtain non-static baglike solutions.

We also asked ourselves whether giving a mass to the fermions would alleviate the problem. The static bag wavefunctions, like their massless counterparts, are not equal to zero at the boundary. A moving boundary, however, requires also in the massive case the boundary conditions equation (12) and hence, as mentioned before, the solution cannot be written as a time-dependent combination of static cavity eigenfunctions. We could not find the general solution in this case, and we conjecture that no solution exists for massive fermions. Our conjecture is based on the fact that equation (12) implies a delicate cancellation of all the Fourier components of the wavefunction. In order to maintain this zero boundary condition at all times, all the Fourier components of the wave should travel at the same speed as that of the moving wall. This is indeed possible in the massless case, if the bag wall expands with the speed of light. For massive fermions however, each Fourier component travels at a different speed, and it may not be possible to satisfy the boundary condition equation (12) at all times.

In this paper we have considered specifically the MIT bag model, which is singular in the sense that no kinetic energy term is associated with the motion of the boundary. However, it is straightforward to see that our result applies without modification to a non-singular model such as the 'Budapest' bag [7], which includes a surface tension besides the volume energy term. This is because the presence of a boundary kinetic term does not modify the linear boundary condition equation (5), which is responsible for the unphysical solution. For the same reason including free gluons inside the bag, i.e. with no quark–gluon interaction term, would not solve the problem. Analogously to the MIT bag, energy conservation would require the 'Budapest' bag to have zero surface tension, besides $B = 0$.

In summary, we have shown that the equations of motion derived from the *dynamical* MIT bag Lagrangian with massless quarks and a spherical boundary admit only one solution corresponding to a bag expanding at the speed of light. This result raises the question of whether the quantization of the theory can provide stable solutions. We infer that a consistent dynamical bag model for absolutely confined fermions must include an interaction term with some other fields at least at the boundary of the domain. For example, in a work to be published elsewhere [14] we shall show that a *dynamical* chiral bag model [4] does admit physically meaningful solutions.

Acknowledgments

We are grateful for the support of a Hong Kong Research Grants Council grant CUHK 4189/97P and a Chinese University Direct Grant (grant no 2060193).

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